The Kakeya Set Conjecture over $\mathbb{Z}/N\mathbb{Z}$ for general N

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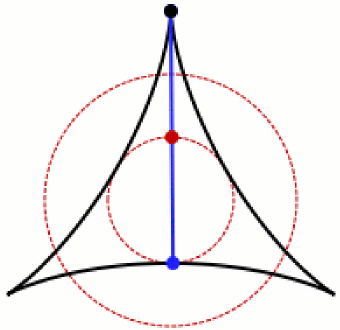
Definition (Kakeya Needle Set)

A Kakeya Needle Set S in $\mathbb{R}^2$ is a set in which a unit line segment can be rotated continuously through 360 degrees returning to its original orientation.



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- What about Hausdorff Dimension? (A set with Hausdorff dimension k needs $\approx (1/\epsilon)^k$ ϵ-balls to cover it)

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- Only known for $n = 2$.

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- Want to lower bound the size of Kakeya Sets.
- First proposed over finite fields [Wolff, 1999] as a simpler version of the Euclidean Kakeya conjecture.
- Also motivated by applications in TCS for constructing randomness mergers and extractors [Dvir and Wigderson, 2011, Dvir, Kopparty, Saraf, and Sudan, 2013].

Kakeya Set bounds over finite-fields

Theorem (Finite-Field Kakeya [Dvir, 2009, Saraf and Sudan, 2008, Dvir, Kopparty, Saraf, and Sudan, 2013, Bukh and Chao, 2021])

Every Kakeya Set S in $(\mathbb{Z}/p\mathbb{Z})^n$,

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$$|S| \gtrapprox N^{n0.59..}$$

using work on the "Sum-Difference conjecture" (also known as the arithmetic Kakeya conjecture) [Bourgain, 1999, Katz and Tao, 1999].

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  using work on the "Sum-Difference conjecture" (also known as the arithmetic Kakeya conjecture) [Bourgain, 1999, Katz and Tao, 1999].
- Positively resolving the Sum-Difference conjecture will also resolve the Euclidean Kakeya conjecture!

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Conjecture (Kakeya Set Conjecture over $\mathbb{Z}/N\mathbb{Z}$ [Hickman and Wright, 2018])

For all $\epsilon > 0$ and $n \in \mathbb{N}$ there exists a constant $C_{n,\epsilon}$ such that any Kakeya Set $S \subset (\mathbb{Z}/N\mathbb{Z})^n$ satisfies

$$|S| \geq C_{n,\epsilon} N^{n-\epsilon}.$$

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- The Kakeya problem over $\mathbb{Z}/p^k\mathbb{Z}$ was suggested in [Ellenberg, Oberlin, and Tao, 2010] as another step towards the Euclidean problem as the ring has "scales".

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- Construction of new codes or improvements for existing ones (Matching Vectors Codes). [Grolmusz, 2000, Dvir, Gopalan, and Yekhanin, 2011]

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- PIR protocols [Dvir and Gopi, 2014]

New results for composites [D and Dvir, 2021]

Theorem (Square-free N [D and Dvir, 2021])

For $N = p_1 \dots p_r$ where p_i are distinct primes, every Kakeya Set S in $(\mathbb{Z}/N\mathbb{Z})^n$ satisfies,

$$|S| \geq \frac{N^n}{2^{nr}}$$

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- Tight up to a factor of 2^r.

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Definition (Matrix $W_{p^k,n}$)

$W_{p^k,n}$ is a matrix whose rows and columns are indexed by points in $(\mathbb{Z}/p^k\mathbb{Z})^n$ with entries,

$$W_{p^k,n}(u,v) = \begin{cases} 1 & \textit{if } \langle u, v\rangle = 0 \\ 0 & \textit{otherwise} \end{cases}$$

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- $W_{p^k,n}$ is a sub-matrix of $V_{p^k,n}$. $V_{p^k,n}$ is a sub-matrix of $W_{p^k,n+1}$.

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- There exist Kakeya Sets in $(\mathbb{Z}/p^k\mathbb{Z})^n$ of size p^{kn}/k^{n-1} [D, 2021].

Resolution of the Kakeya Set Conjecture for general N

Theorem (General $\mathbb{Z}/N\mathbb{Z}$ bound [D, 2021])

Every Kakeya Set in $(\mathbb{Z}/N\mathbb{Z})^n$ for $N = p_1^{k_1} \dots p_r^{k_r}$ has size at least

$$\frac{N^n}{\left(\prod_{i=1}^{r} 2^n (k_i + \log_p(n))^n\right)} \geq C_{n,\epsilon} N^{n-\epsilon}.$$

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- As $p_i \to \infty, \forall i = \{1, \dots, r\}$ the constant can be improved to $(\prod_{i=1}^{r}(k_i + 1))^{-n}$ recovering the square-free N bound from [D and Dvir, 2021].

Talk Overview

1. The Polynomial Method over $\mathbb{Z}/p\mathbb{Z}$
2. "New" Proof for $\mathbb{Z}/p\mathbb{Z}$
3. Proof for $\mathbb{Z}/pq\mathbb{Z}$
4. Proof for $\mathbb{Z}/p^k\mathbb{Z}$

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- $\exists f \neq 0, f \in \mathbb{F}_p[x_1, \ldots, x_n]$ of degree $D \leq p-1$ which vanishes on S.
- For every direction $u \in \mathbb{F}_p^n$, f vanishes on some line $L_u = \{x + \lambda u | \lambda \in \mathbb{F}_p\}$ contained in S.
- $f(x + \lambda u)$ is a uni-variate polynomial in λ of degree $D \leq p-1$ with p zeros which means it is identically 0.

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- f_D is a non-zero homogenous polynomial of degree $D \leq p - 1$ which vanishes on all of $\mathbb{F}_p^n$.
- Contradiction (due to the DeMillo-Lipton-Schwartz–Zippel lemma).

$\square$

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- $(x^p - x)(x^q - x)$ vanishes over all of $\mathbb{Z}/pq\mathbb{Z}$.
- If we try to adapt the proof strategy above for $N = p^2, pq$ we will get a lower bound of $\binom{p+n-1}{n} \approx p^n \approx N^{0.5n}$.

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Definition (Line matrix M_S of a Kakeya Set S)

The line matrix M_S of S is a matrix where the u'th row is the indicator vector $\mathbb{1}_{L_u}$ of L_u in direction u which is contained in S.

- WLOG we assume that $S = \bigcup_{u\in(\mathbb{Z}/N\mathbb{Z})^n} L_u$ where L_u is a line in direction u.

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For any field $\mathbb{F}$, $|S| \geq \text{rank}_{\mathbb{F}} M_S \geq |S|/N$.

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$\mathbf{|S| \geq rank_{\mathbb{F}} M_S}$**:** The non-zero columns of M_S correspond to points in S.
$\mathbf{rank_{\mathbb{F}} M_S \geq |S|/N}$**:** Iteratively pick lines in S such that every new line you pick has a point not covered by the earlier lines. These at least $|S|/N$ many lines will give linearly independent rows. □

Idea

To lower bound the rank of M_S find a matrix A such that

$$M_S \cdot A = B$$

and B is a matrix independent of S.

“New” Proof for $\mathbb{Z}/p\mathbb{Z}$ [D and Dvir, 2021]

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To lower bound the rank of M_S find a matrix A such that

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and B is a matrix independent of S.

- $A = W_{p,n}$ works!

“New” Proof for $\mathbb{Z}/p\mathbb{Z}$ [D and Dvir, 2021]

Rank lower bound for M_S

Claim

In the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, for a line $L = \{a + \lambda u | \lambda \in \mathbb{F}_p\}$ we have

$$\mathbb{1}_L \cdot W_{p,n} = \mathbb{1}_{\overline{H}_u},$$

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- $$M_S \cdot W_{p,n} = \mathbf{1} - W_{p,n}$$

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Theorem ($\mathbb{F}_p$-rank of $W_{p,n}$ [Goethals and Delsarte, 1968, MacWilliams and Mann, 1968, Smith, 1969])

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- Gives us a Kakeya size lower bound of

$$|S| \geq \operatorname{rank}_{\mathbb{F}_p} W_{p,n} - 1 \geq \binom{p+n-2}{n-1} - 2 \geq \frac{p^{n-1}}{n!}.$$

$$\mathrm{EVAL}_{p,n} = \begin{array}{cc} & m \\ x & \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & m(x) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \end{array} = \begin{bmatrix} \cdots & | & \cdots \\ \cdots & m(\mathbb{F}_p^n) & \cdots \\ \cdots & | & \cdots \end{bmatrix}$$

where $m \in \mathbb{F}_p[x_1, \ldots, x_n]$ is a monomial of degree $p-1$ and $x \in \mathbb{F}_p^n$.

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- M_S acts as a "decoder".
- We can extend M_S to use the evaluation of derivatives to get stronger bounds.

Kakeya Sets in $\mathbb{Z}/pq\mathbb{Z}$

Some facts about $(\mathbb{Z}/pq\mathbb{Z})^n$

- By the Chinese remainder theorem we know that $(\mathbb{Z}/pq\mathbb{Z})^n \cong \mathbb{F}_p^n \times \mathbb{F}_q^n$.

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$$L_u = L_p(u) \times L_q(u)$$

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- Note, while the product $S_p \times S_q$ of two Kakeya Sets $S_p \subseteq \mathbb{F}_p^n$ and $S_q \subseteq \mathbb{F}_q^n$ is a Kakeya Set the converse is not true.

Kakeya Sets in $\mathbb{Z}/pq\mathbb{Z}$ [D and Dvir, 2021]

Theorem (Simple Kakeya Set bounds for $\mathbb{Z}/pq\mathbb{Z}$ [D and Dvir, 2021])

Every Kakeya Set S in $(\mathbb{Z}/pq\mathbb{Z})^n$ has size at least,

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  where $1 \leq i \leq p^{n-1}/n!$
- We saw earlier that $\mathbb{1}_{\overline{H}_{u_p}}$ for $u_p \in \mathbb{F}_p^n$ has rank at least $p^{n-1}/n!$.

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Questions?

Proof Strategy

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- "(mod p)" map doesn't increase rank.

The rings T_k and $\overline{T}_k$

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The map ψ which maps ζ to 1, $\mathbb{Z}$ to $\mathbb{F}_p$ (via the mod p map) and z to z is a ring homomorphism from T_k onto $\overline{T}_k$.

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- Note, $\phi(1) = 0 \pmod p$,equivalently $x - 1$ divides $\phi(x)$ in $\mathbb{F}_p$.

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Definition (Matrix $V_{p^k,n}$)

$V_{p^k,n}$ is a $p^{kn} \times p^{kn}$ matrix with entries in $\overline{T}_k = \mathbb{F}_p[z]/\langle z^{p^k} - 1\rangle$ whose entries are,

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- Lower bounding the number of non-zero diagonal elements of U gives the rank bound.

Decoding evaluations on the complex torus [D, 2021]

- For $f \in \mathbb{Z}[y_1, \ldots, y_n]$ the $f(z^u) \in \overline{T}_k$ is in $\psi(\text{span}\{f(\zeta^{L_u})\})$ where L_u is a line in direction u.

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Lemma (Decoding evaluations along lines on the $\mathbb{C}$ torus [D, 2021])

Given $L_u = \{a + \lambda u | \lambda \in \mathbb{Z}/p^k\mathbb{Z}\}$ there exists $c_x \in \frac{\mathbb{Q}(\zeta)[z]}{\langle z^{p^k} - 1 \rangle}, x \in L_u$ such that,

$$\psi\left(\sum_{x \in L_u} c_x f(\zeta^x)\right) = f(z^u),$$

for all polynomials $f \in \mathbb{Z}[y_1, \ldots, y_n]$.

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There exists constants $c_\lambda \in \mathbb{Q}(\zeta)[z]/\langle z^{p^k} - 1 \rangle$ for $\lambda = 1, \dots, p^k$ such that

$$\sum_{\lambda=1}^{p^k} c_\lambda m_v(\zeta^{\lambda u}) = z^{\langle u,v \rangle} = m_v(z^u) \in \frac{\mathbb{Z}(\zeta)[z]}{\langle z^{p^k} - 1 \rangle} = T_k.$$

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- Applying ψ gives us,
$$\psi\left(\sum_{\lambda=1}^{p^k} c_\lambda m_v(\zeta^{a+\lambda u})\right) = m_v(z^u) \in \overline{T}_k.$$

$\square$

- Can be extended to decode with derivatives at fewer points.

Decoding evaluations on the complex torus

Corollary (Decode $V_{p^k,n}(u)$ from $E_{p^k,n}(\zeta^{L_u})$)

For a line $L_u = \{a + \lambda u | \lambda \in \mathbb{Z}/p^k\mathbb{Z}\}$ we can find a row vector C_u indexed by points in $(\mathbb{Z}/p^k\mathbb{Z})^n$ such that,

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- The matrix C with rows C_u for each line L_u in S is the required decoder matrix.